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Positive Solutions of Complementary Lidstone Boundary Value Problems

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Abstract. We consider the following complementary Lidstone boundary value problem

$$
\begin{aligned}
&(-1)^m y^{(2m+1)}(t) = F(t, y(t), y'(t)), \quad t \in [0,1]\\
&y(0) = 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, \quad 1 \le k \le m.
\end{aligned}
$$

The nonlinear term F depends on y' and this derivative dependence is seldom investigated in the literature. Using a variety of fixed point theorems, we establish the existence of one or more positive solutions for the boundary value problem. Examples are also included to illustrate the results obtained.

Keywords: Derivative dependence, positive solutions, complementary Lidstone boundary value problems.

AMS Subject Classification: 34B15, 34B18.

1 Introduction

In this paper we shall consider the *complementary Lidstone* boundary value problem

$$
\begin{aligned}
&(-1)^m y^{(2m+1)}(t) = F(t, y(t), y'(t)), \quad t \in [0,1]\\
&y(0) = 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, \quad 1 \le k \le m
\end{aligned}
\tag{1.1}
$$

where $m \ge 1$ and F is continuous at least in the interior of the domain of interest. It is noted that the nonlinear term F involves y', a derivative of the dependent variable. This case is seldom studied in the literature and most research papers on boundary value problems consider nonlinear terms that involve y only.

The *complementary Lidstone* interpolation and boundary value problems are very recently introduced in [17], and drawn on by Agarwal et. al. in [3, 9] where they consider an $(2m+1)$th order differential equation together with boundary data at the odd order derivatives

$$
y(0) = a_0, \quad y^{(2k-1)}(0) = a_k, \quad y^{(2k-1)}(1) = b_k, \quad 1 \le k \le m. \tag{1.2}
$$

The boundary conditions (1.2) are known as *complementary Lidstone* boundary conditions, they naturally complement the *Lidstone* boundary conditions [4, 6, 19, 31] which involve even order

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derivatives. To be precise, the *Lidstone* boundary value problem comprises an $2m$th order differential equation and the *Lidstone* boundary conditions

$$y^{(2k)}(0) = a_k, \quad y^{(2k)}(1) = b_k, \quad 0 \le k \le m-1. \tag{1.3}$$

There is a vast literature on Lidstone interpolation and boundary value problems. In fact, the Lidstone interpolation was first introduced by Lidstone [26] in 1929 and further characterized in the work of [13, 14, 28, 29, 32, 33, 34, 35]. More recent research on Lidstone interpolation as well as Lidstone spline can be found in [7, 8, 16, 17, 18, 36, 37, 38]. On the other hand, the Lidstone boundary value problems and several of its particular cases have been the subject matter of numerous investigations, see [1, 2, 4, 5, 8, 11, 12, 15, 20, 21, 22, 23, 24, 27, 30, 39] and the references cited therein. It is noted that in most of these works the nonlinear terms considered do *not* involve derivatives of the dependent variable, only a handful of papers [20, 21, 24, 27] tackle nonlinear terms that involve even order derivatives. In the present work, our study of the *complementary Lidstone* boundary value problem (1.1) where F depends on a *derivative* certainly extends and complements the rich literature on boundary value problems and in particular on Lidstone boundary value problems. The literature on *complementary Lidstone* boundary value problems pales in comparison with that of *Lidstone* boundary value problems, in a recent work [10] the eigenvalue problem of *complementary Lidstone* boundary value problem is discussed.

The focus of this paper is on the existence of a positive solution of (1.1). By a *positive solution* y of (1.1), we mean a nontrivial $y \in C[0,1]$ satisfying (1.1) and $y(t) \ge 0$ for $t \in [0,1]$. By using a variety of fixed point theorems, we begin with the establishment of the existence of a solution (not necessary positive), and proceed to develop the existence of a nontrivial positive solution, two nontrivial positive solutions, and multiple nontrivial positive solutions. The usefulness of the results obtained are then illustrated by some examples.

2 Preliminaries

In this section we shall state the fixed point theorems and some inequalities for certain Green's function which are needed later. The first theorem is known as the *Leray-Schauder alternative* and the second is usually called *Krasnosel'skii's fixed point theorem in a cone.*

Theorem 2.1. [2] Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \overline{U} \to E$ is a continuous and compact map. Then either

(a) S has a fixed point in $\overline{U}$, or

(b) there exists $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda Sx$.

Theorem 2.2. [25] Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a completely continuous operator such that, either

(a) $\|Sx\| \le \|x\|$, $x \in C \cap \partial\Omega_1$, and $\|Sx\| \ge \|x\|$, $x \in C \cap \partial\Omega_2$, or

(b) $\|Sx\| \geq \|x\|$, $x \in C \cap \partial\Omega_1$, and $\|Sx\| \leq \|x\|$, $x \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To tackle the *complementary Lidstone* boundary value problem (1.1), let us review certain attributes of the *Lidstone* boundary value problem. Let $g_m(t,s)$ be the Green's function of the Lidstone boundary value problem

$$\begin{aligned} x^{(2m)}(t) = 0, \quad t \in [0,1] \\ x^{(2k)}(0) = x^{(2k)}(1) = 0, \quad 0 \leq k \leq m-1. \end{aligned} \tag{2.1}$$

The Green's function $g_m(t,s)$ can be expressed as [4, 6]

$$g_m(t,s) = \int_0^1 g(t,u) g_{m-1}(u,s) du \tag{2.2}$$

where

$$g_1(t,s) = g(t,s) = \begin{cases} t(s-1), & 0 \leq t \leq s \leq 1 \\ s(t-1), & 0 \leq s \leq t \leq 1. \end{cases}$$

Further, it is known that

$$|g_m(t,s)| = (-1)^m g_m(t,s) \quad \text{and} \quad g_m(t,s) = g_m(s,t), \quad (t,s) \in (0,1) \times (0,1). \tag{2.3}$$

The following two lemmas give the upper and lower bounds of $|g_m(t,s)|$, they play an important role in subsequent development. We remark that the bounds in the two lemmas are *sharper* than those given in the literature [4, 6, 27, 39].

Lemma 2.1. [10] For $(t,s) \in [0,1] \times [0,1]$, we have

$$|g_m(t,s)| \leq \frac{1}{\pi^{2m-1}} \sin \pi s.$$

Lemma 2.2. [10] Let $\delta \in \left(0, \frac{1}{2}\right)$ be given. For $(t,s) \in [\delta, 1-\delta] \times [0,1]$, we have

$$|g_m(t,s)| \geq \frac{2\delta}{\pi^{2m}} \sin \pi s.$$

3 Existence of Positive Solutions

To tackle (1.1) we first consider the initial value problem

$$\begin{aligned} y'(t) = x(t), \quad t \in [0,1] \\ y(0) = 0 \end{aligned} \tag{3.1}$$

whose solution is simply

$$y(t) = \int_0^t x(s) ds. \tag{3.2}$$

Taking into account (3.1) and (3.2), the *complementary Lidstone* boundary value problem (1.1) reduces to the *Lidstone* boundary value problem

$$\begin{aligned} &(-1)^m x^{(2m)}(t) = F\left(t, \int_0^t x(s)ds, x(t)\right), \quad t \in [0,1] \\ &x^{(2k-2)}(0) = x^{(2k-2)}(1) = 0, \quad 1 \le k \le m. \end{aligned} \tag{3.3}$$

If (3.3) has a solution x^*, then by virtue of (3.2),

$$y^*(t) = \int_0^t x^*(s)ds \tag{3.4}$$

is a solution of (1.1). Hence, the existence of a solution of the *complementary Lidstone* boundary value problem (1.1) follows from the existence of a solution of the *Lidstone* boundary value problem (3.3). It is clear from (3.4) that $\|y^*\| \le \|x^*\|$, moreover if x^* is positive, so is y^*. With the tools in Section 2 and a technique to handle the nonlinear term F, we shall study the boundary value problem (1.1) via (3.3).

Let the Banach space $B = C[0,1]$ be equipped with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$ for $x \in B$. Define the operator $S : C[0,1] \to C[0,1]$ by

$$\begin{aligned} Sx(t) &= \int_0^1 (-1)^m g_m(t,s) F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \\ &= \int_0^1 |g_m(t,s)| F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds, \quad t \in [0,1] \end{aligned} \tag{3.5}$$

where $g_m(t,s)$ is the Green's function given in (2.2). A fixed point x^* of the operator S is clearly a solution of the boundary value problem (3.3), and as seen earlier $y^*(t) = \int_0^t x^*(s)ds$ is a solution of (1.1).

For easy reference, we list below the conditions that are used later. In these conditions, the number $\delta \in \left(0, \frac{1}{2}\right)$ is fixed and the sets K, $\tilde{K}$ are defined by

$$\tilde{K} = \{x \in B \mid x(t) \ge 0,\ t \in [0,1]\}$$

and

$$K = \{x \in \tilde{K} \mid x(t) > 0 \text{ on some subset of } [0,1] \text{ of positive measure}\}.$$

(C1) F is continuous on $[0,1] \times \tilde{K} \times \tilde{K}$, with

$$F(t,u,v) \ge 0, \quad (t,u,v) \in [0,1] \times \tilde{K} \times \tilde{K} \qquad \text{and} \qquad F(t,u,v) > 0, \quad (t,u) \in [0,1] \times K \times K.$$

(C2) There exist continuous functions β and f with $\beta : [0,1] \to [0,\infty)$, $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ and f is nondecreasing in each of its arguments, such that

$$F(t,u,v) \le \beta(t) f(u,v), \quad (t,u,v) \in [0,1] \times \tilde{K} \times \tilde{K}.$$

(C3) There exists $a > 0$ such that

$$a > Mf(a,a)$$

where $M = \sup_{t\in[0,1]} \int_0^1 |g_m(t,s)|\beta(s)ds$.

(C4) There exists a continuous function α with $\alpha : \left[\frac{1}{2}, 1-\delta\right] \to (0, \infty)$, such that

$$F(t,u,v) \geq \alpha(t) f(u,v), \quad (t,u,v) \in \left[\frac{1}{2}, 1-\delta\right] \times K \times K.$$

(C5) There exists $b > 0$ such that

$$b \leq N f\left(\gamma b \left(\frac{1}{2} - \delta\right), \gamma b\right)$$

where $\gamma = \frac{2\delta}{\pi}$ and $N = \sup_{t\in[0,1]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) ds$.

Our first result is an existence criterion for a solution (need *not* be positive).

Theorem 3.1. Let $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there exists a constant ρ, independent of λ, such that $\|x\| \neq \rho$ for any solution $x \in C[0,1]$ of the equation

$$x(t) = \lambda \int_0^1 |g_m(t,s)| F\left(s, \int_0^s x(\tau) d\tau, x(s)\right) ds, \ t \in [0,1] \qquad (3.6)_\lambda$$

where $0 < \lambda < 1$. Then, (1.1) has at least one solution $y^* \in C[0,1]$ such that $\|y^*\| \leq \rho$.

Proof. Clearly, a solution of $(3.6)_\lambda$ is a fixed point of the equation $x = \lambda S x$ where S is defined in (3.5). Using the Arzelà-Ascoli theorem, we see that S is continuous and completely continuous. Now, in the context of Theorem 2.1, let $U = \{x \in B \mid \|x\| < \rho\}$. Since $\|x\| \neq \rho$, where x is any solution of $(3.6)_\lambda$, we cannot have conclusion (b) of Theorem 2.1, hence conclusion (a) of Theorem 2.1 must hold, i.e., (3.3) has a solution $x^* \in \overline{U}$ with $\|x^*\| \leq \rho$. From (3.4), it is clear that (1.1) has a solution $y^*(t) = \int_0^t x^*(s) ds$ with $\|y^*\| \leq \|x^*\| \leq \rho$. $\square$

The next result employs Theorem 3.1 to give the existence of a *positive* solution.

Theorem 3.2. Let (C1)–(C3) hold. Then, (1.1) has a positive solution $y^* \in C[0,1]$ such that $\|y^*\| < a$, i.e., $0 \leq y^*(t) < a, \ t \in [0,1]$.

Proof. To apply Theorem 3.1, we consider the equation

$$x(t) = \int_0^1 |g_m(t,s)| \hat{F}\left(s, \int_0^s x(\tau) d\tau, x(s)\right) ds, \ t \in [0,1] \qquad (3.7)$$

where $\hat{F} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\hat{F}(t,u,v) = F(t, |u|, |v|). \qquad (3.8)$$

Noting (C1) we see that the function $\hat{F}$ is well defined and is continuous.

We shall show that (3.7) has a solution. To proceed, we shall consider the equation

$$x(t) = \lambda \int_0^1 |g_m(t,s)| \hat{F}\left(s, \int_0^s x(\tau) d\tau, x(s)\right) ds, \ t \in [0,1] \qquad (3.9)_\lambda$$

where $0 < \lambda < 1$, and show that any solution $x \in C[0,1]$ of $(3.9)_\lambda$ satisfies $\|x\| \neq a$. Then it follows from the proof of Theorem 3.1 that (3.7) has a solution.

Let $x \in C[0,1]$ be any solution $(3.9)_\lambda$. Using (3.8) and (C1) we get

$$\begin{aligned} x(t) &= \lambda \int_0^1 |g_m(t,s)| \hat{F}\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \\ &= \lambda \int_0^1 |g_m(t,s)| F\left(s, \left|\int_0^s x(\tau)d\tau\right|, |x(s)|\right) ds \ \geq \ 0, \quad t \in [0,1]. \end{aligned}$$

Thus, x is a *positive* solution.

Applying (C2) and (C3) successively, we find for $t \in [0,1]$,

$$\begin{aligned} |x(t)| \ = \ x(t) \ &\leq \ \int_0^1 |g_m(t,s)| F\left(s, \left|\int_0^s x(\tau)d\tau\right|, |x(s)|\right) ds \\ &\leq \ \int_0^1 |g_m(t,s)| \beta(s) f\left(\left|\int_0^s x(\tau)d\tau\right|, |x(s)|\right) ds \\ &\leq \ \int_0^1 |g_m(t,s)| \beta(s) f\left(\int_0^1 \|x\| d\tau, \|x\|\right) ds \\ &= \ \int_0^1 |g_m(t,s)| \beta(s) ds \cdot f(\|x\|, \|x\|). \end{aligned}$$

Taking supremum both sides yields

$$\|x\| \leq M f(\|x\|, \|x\|). \tag{3.10}$$

Comparing (3.10) and (C3), we conclude that $\|x\| \neq a$.

It now follows from the proof of Theorem 3.1 that (3.7) has a solution $x^* \in C[0,1]$ with $\|x^*\| \leq a$. Using a similar argument as above, it can be easily seen that x^* is a *positive* solution and $\|x^*\| \neq a$. Hence, $\|x^*\| < a$.

Finally, we shall show that x^* is actually a solution of (3.3). In fact, using (3.8) and the positivity of x^*, we obtain for $t \in [0,1]$,

$$\begin{aligned} x^*(t) &= \int_0^1 |g_m(t,s)| \hat{F}\left(s, \int_0^s x^*(\tau)d\tau, x^*(s)\right) ds \\ &= \int_0^1 |g_m(t,s)| F\left(s, \left|\int_0^s x^*(\tau)d\tau\right|, |x^*(s)|\right) ds \\ &= \int_0^1 |g_m(t,s)| F\left(s, \int_0^s x^*(\tau)d\tau, x^*(s)\right) ds. \end{aligned}$$

Hence, x^* is a positive solution of (3.3) with $\|x^*\| < a$. Noting (3.4), $y^*(t) = \int_0^t x^*(s)ds$ is a positive solution of (1.1) with $\|y^*\| \leq \|x^*\| < a$. $\square$

Remark 3.1. We note that the last inequality in (C1), viz,

$$F(t,u,v) > 0, \quad (t,u,v) \in [0,1] \times K \times K$$

is *not* needed in Theorem 3.2.

Theorem 3.2 provides the existence of a positive solution which may be trivial. Our next result guarantees the existence of a <u>*nontrivial positive*</u> solution.

Theorem 3.3. Let (C1)–(C5) hold. Then, (1.1) has a nontrivial positive solution $y^* \in C[0,1]$ such that

(a) $\|y^*\| \le b$ and $y^*(t) > \gamma a(t-\delta)$ for $t \in [\delta, 1-\delta]$, if $a < b$;

(b) $\|y^*\| < a$ and $y^*(t) \ge \gamma b(t-\delta)$ for $t \in [\delta, 1-\delta]$, if $a > b$.

Proof. We shall employ Theorem 2.2. To begin, note that the operator $S : C[0,1] \to C[0,1]$ is continuous and completely continuous.

Next, we define a cone $C \subset B$ by

$$C = \left\{ x \in B \;\middle|\; x(t) \ge 0 \text{ for } t \in [0,1], \text{ and } \min_{t\in[\delta,1-\delta]} x(t) \ge \gamma \|x\| \right\} \tag{3.11}$$

where $\gamma = \frac{2\delta}{\pi}$ (< 1). Note that $C \subseteq \tilde{K}$. We shall show that S maps C into C. Let $x \in C$. Noting (C1), we obtain

$$Sx(t) = \int_0^1 |g_m(t,s)| F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \ge 0, \;\; t \in [0,1]. \tag{3.12}$$

Next, using (3.12) and Lemma 2.1, we have for $t \in [0,1]$,

$$|Sx(t)| = Sx(t) \le \int_0^1 \frac{1}{\pi^{2m-1}} \; F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) \sin \pi s ds$$

which leads to

$$\|Sx\| \le \int_0^1 \frac{1}{\pi^{2m-1}} \; F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) \sin \pi s ds. \tag{3.13}$$

On the other hand, for $t \in [\delta, 1-\delta]$ we use Lemma 2.2 and (3.13) to get

$$Sx(t) \ge \int_0^1 \frac{2\delta}{\pi^{2m}} \; F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) \sin \pi s ds \ge \frac{2\delta}{\pi} \; \|Sx\|.$$

It follows that

$$\min_{t\in[\delta,1-\delta]} Sx(t) \ge \gamma \|Sx\|. \tag{3.14}$$

Having established (3.12) and (3.14), we have shown that $S(C) \subseteq C$.

Let

$$\Omega_a = \{x \in B \mid \|x\| < a\} \qquad \text{and} \qquad \Omega_b = \{x \in B \mid \|x\| < b\}.$$

We shall show that (i) $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_a$, and (ii) $\|Sx\| \ge \|x\|$ for $x \in C \cap \partial\Omega_b$.

To verify (i), let $x \in C \cap \partial\Omega_a$. Then, $\|x\| = a$. Using (C2), we get for $t \in [0,1]$,

$$|Sx(t)| = Sx(t) \le \int_0^1 |g_m(t,s)|\beta(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \le \int_0^1 |g_m(t,s)|\beta(s) f\left(\int_0^1 a \; d\tau, a\right) ds.$$

Taking supremum and applying (C3) then gives

$$\|Sx\| \le M f(a,a) < a = \|x\|. \tag{3.15}$$

Next, to prove (ii), let $x \in C \cap \partial\Omega_b$. So $\|x\| = b$. Noting (C4), we find for $t \in [0,1]$,

$$\begin{aligned}
|Sx(t)| &\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) f\left(\int_\delta^{\frac{1}{2}} x(\tau)d\tau, x(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) f\left(\int_\delta^{\frac{1}{2}} \gamma b\ d\tau, \gamma b\right) ds.
\end{aligned}$$

Taking supremum both sides and using (C5), we obtain

$$\|Sx\| \geq N f\left(\gamma b\left(\frac{1}{2} - \delta\right), \gamma b\right) \geq b = \|x\|. \tag{3.16}$$

Having established (i) and (ii), it follows from Theorem 2.2 that S has a fixed point $x^* \in C \cap \left(\overline{\Omega}_{\max\{a,b\}} \backslash \Omega_{\min\{a,b\}}\right)$. Thus, $\min\{a,b\} \leq \|x^*\| \leq \max\{a,b\}$. Using a similar argument as in the first part of the proof of Theorem 3.2, we see that $\|x^*\| \neq a$. Hence, we obtain

$$a < \|x^*\| \leq b \ \text{ if } \ a < b \qquad \text{and} \qquad b \leq \|x^*\| < a \ \text{ if } \ a > b. \tag{3.17}$$

Coupling (3.17) with the fact $x^* \in C$ gives

$$\min_{t\in[\delta,1-\delta]} x^*(t) \geq \gamma\|x^*\| \begin{cases} > \gamma a, & \text{if } \ a < b \\ \geq \gamma b, & \text{if } \ a > b. \end{cases}$$

Now from (3.4), a positive solution of (1.1) is $y^*(t) = \int_0^t x^*(s)ds$. In view of (3.17), it is clear that

$$\|y^*\| \leq \|x^*\| \begin{cases} \leq b, & \text{if } \ a < b \\ < a, & \text{if } \ a > b. \end{cases}$$

Moreover, we have for $t \in [\delta, 1-\delta]$,

$$y^*(t) = \int_0^t x^*(s)ds \geq \int_\delta^t x^*(s)ds \geq \int_\delta^t \gamma\|x^*\|ds = \gamma\|x^*\|(t-\delta). \tag{3.18}$$

Hence, noting (3.17) we get for $t \in [\delta, 1-\delta]$,

$$y^*(t) > \gamma a(t-\delta) \ \text{ if } \ a < b \qquad \text{and} \qquad y^*(t) \geq \gamma b(t-\delta) \ \text{ if } \ a > b.$$

The proof is complete. $\square$

Remark 3.2. The conditions (C4) and (C5) in Theorem 3.3 may be replaced by the following:

(C4)$'$ there exists a continuous function α_0 with $\alpha_0 : [1-\delta, 1] \to (0,\infty)$, such that

$$F(t,u,v) \geq \alpha_0(t) f(u,v), \quad (t,u,v) \in [1-\delta, 1] \times K \times K;$$

(C5)$'$ there exists $b > 0$ such that
$$b \le N_0 f\left(\gamma b(1-2\delta), \gamma b\right)$$
where $\gamma = \frac{2\delta}{\pi}$ and $N_0 = \sup_{t\in[0,1]} \int_{1-\delta}^1 |g_m(t,s)|\alpha_0(s)ds$.

Indeed, in the proof of Theorem 3.3, to show (ii) $\|Sx\| \ge \|x\|$ for $x \in C \cap \partial\Omega_b$, using (C4)$'$ we find for $x \in C \cap \partial\Omega_b$ and $t \in [0,1]$,

$$\begin{aligned} |Sx(t)| &\ge \int_{1-\delta}^1 |g_m(t,s)| F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\ge \int_{1-\delta}^1 |g_m(t,s)|\alpha_0(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\ge \int_{1-\delta}^1 |g_m(t,s)|\alpha_0(s) f\left(\int_\delta^{1-\delta} x(\tau)d\tau, x(s)\right) ds \\ &\ge \int_{1-\delta}^1 |g_m(t,s)|\alpha_0(s) f\left(\int_\delta^{1-\delta} \gamma b\ d\tau, \gamma b\right) ds. \end{aligned}$$

Now, taking supremum both sides and using (C5)$'$ yields

$$\|Sx\| \ge N_0 f\left(\gamma b(1-2\delta), \gamma b\right) \ge b = \|x\|.$$

Remark 3.3. The computation of the constants M, N and N_0 in (C3), (C5) and (C5)$'$ can be avoided by using Lemmas 2.1 and 2.2, the tradeoff is we obtain *stricter* inequalities. Indeed, using Lemma 2.1 we have

$$M = \sup_{t\in[0,1]} \int_0^1 |g_m(t,s)|\beta(s)ds \le \int_0^1 \frac{1}{\pi^{2m-1}}\ \beta(s)\sin\pi s ds$$

and so (C3) is satisfied provided

$$a > f(a,a) \int_0^1 \frac{1}{\pi^{2m-1}}\ \beta(s)\sin\pi s ds, \tag{3.19}$$

which is a *stronger* condition to fulfill. On the other hand, in view of Lemma 2.2, we have

$$N = \sup_{t\in[0,1]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)|\alpha(s)ds \ge \sup_{t\in[\delta,1-\delta]} \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)|\alpha(s)ds \ge \int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\ \alpha(s)\sin\pi s ds$$

and so (C5) is fulfilled if we impose the *stricter* inequality

$$b \le f\left(\gamma b\left(\frac{1}{2} - \delta\right), \gamma b\right) \int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\ \alpha(s)\sin\pi s ds. \tag{3.20}$$

Similarly, (C5)$'$ is satisfied provided we have the *stricter* inequality

$$b \le f\left(\gamma b(1-2\delta), \gamma b\right) \int_{1-\delta}^1 \frac{2\delta}{\pi^{2m}}\ \alpha_0(s)\sin\pi s ds. \tag{3.21}$$

The next result gives the existence of *two positive* solutions.

Theorem 3.4. Let (C1)–(C5) hold with $a < b$. Then, (1.1) has (at least) two positive solutions $y_1, y_2 \in C[0,1]$ such that

$$0 \le \|y_1\| < a, \quad \|y_2\| \le b; \quad y_2(t) > \gamma a(t-\delta), \quad t \in [\delta, 1-\delta].$$

Proof. From the proof of Theorems 3.2 and 3.3 (see (3.17)), we conclude that (3.3) has two positive solutions $x_1, x_2 \in C[0,1]$ such that

$$0 \le \|x_1\| < a < \|x_2\| \le b. \tag{3.22}$$

Noting (3.4) and (3.18), it follows that (1.1) has two positive solutions $y_1, y_2 \in C[0,1]$ such that for $i = 1, 2$,

$$\|y_i\| \le \|x_i\| \qquad \text{and} \qquad y_2(t) \ge \gamma\|x_2\|(t-\delta),\ t \in [\delta, 1-\delta]. \tag{3.23}$$

Using (3.22) in (3.23), the conclusion is immediate. $\square$

In Theorem 3.4 it is possible to have $\|y_1\| = 0$. Our next result guarantees the existence of *two nontrivial positive* solutions.

Theorem 3.5. Let (C1)–(C5) and $(C5)|_{b=\tilde{b}}$ hold, where $0 < \tilde{b} < a < b$. Then, (1.1) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that

$$\|y_1\| < a, \quad \|y_2\| \le b; \quad y_1(t) \ge \gamma\tilde{b}(t-\delta), \quad y_2(t) > \gamma a(t-\delta), \quad t \in [\delta, 1-\delta].$$

Proof. From the proof of Theorem 3.3 (see (3.17)), it is clear that (3.3) has two positive solutions $x_1, x_2 \in C[0,1]$ such that

$$0 < \tilde{b} \le \|x_1\| < a < \|x_2\| \le b. \tag{3.24}$$

Noting (3.4), (3.18) and (3.24), the conclusion is clear. $\square$

The next two results also guarantee the existence of *two nontrivial positive* solutions. Unlike Theorem 3.5 which requires *both* (C3) and (C5), these results use *either* (C3) *or* (C5) together with conditions on f_0 and f_∞, where

$$f_0 = \lim_{u\to 0+,\ v\to 0+} \frac{f(u,v)}{v} \qquad \text{and} \qquad f_\infty = \lim_{u\to\infty,\ v\to\infty} \frac{f(u,v)}{v}.$$

Theorem 3.6. Let (C1)–(C4) hold and $0 < \int_{\frac{1}{2}}^{1-\delta} \alpha(s)\sin\pi s ds < \infty$.

(a) If $f_0 = \infty$, then (1.1) has a nontrivial positive solution $y_1 \in C[0,1]$ such that $0 < \|y_1\| < a$.

(b) If $f_\infty = \infty$, then (1.1) has a nontrivial positive solution $y_2 \in C[0,1]$ such that $y_2(t) > \gamma a(t-\delta)$ for $t \in [\delta, 1-\delta]$.

(c) If $f_0 = f_\infty = \infty$, then (1.1) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that

$$0 < \|y_1\| < a \qquad \text{and} \qquad y_2(t) > \gamma a(t-\delta),\ t \in [\delta, 1-\delta].$$

Proof. We shall apply Theorem 2.2 with the cone C defined in (3.11).

(a) Let

$$A = \left[\gamma \int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\, \alpha(s) \sin \pi s ds\right]^{-1}. \tag{3.25}$$

Since $f_0 = \infty$, there exists $0 < r < a$ such that

$$f(u, v) \geq Av, \quad 0 < u \leq r,\ 0 < v \leq r. \tag{3.26}$$

Let $\Omega_r = \{x \in B \mid \|x\| < r\}$. We shall show that $\|Sx\| \geq \|x\|$ for $x \in C \cap \partial\Omega_r$. To proceed, let $x \in C \cap \partial\Omega_r$. So $\|x\| = r$. Applying (C4), Lemma 2.2, (3.26) and (3.25) successively, we get for $t \in [\delta, 1-\delta]$,

$$\begin{aligned} |Sx(t)| &\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| F\left(s, \int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)| \alpha(s) A x(s) ds \\ &\geq \int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\, \alpha(s) A\gamma \|x\| \sin \pi s ds \ = \ \|x\|. \end{aligned}$$

It follows that $\|Sx\| \geq \|x\|$ for $x \in C \cap \partial\Omega_r$.

Next, let $\Omega_a = \{x \in B \mid \|x\| < a\}$. For $x \in C \cap \partial\Omega_a$, using (C2) and (C3) as in the proof of Theorem 3.3, we obtain (3.15). Hence, $\|Sx\| \leq \|x\|$ for $x \in C \cap \partial\Omega_a$.

It now follows from Theorem 2.2 that S has a fixed point $x_1 \in C \cap (\bar{\Omega}_a \backslash \Omega_r)$ such that $r \leq \|x_1\| \leq a$. Using a similar argument as in the first part of the proof of Theorem 3.2, we see that $\|x_1\| \neq a$. Hence, we obtain $r \leq \|x_1\| < a$. From (3.4), we have $y_1(t) = \int_0^t x_1(s)ds$ is a positive solution of (1.1) with $0 < \|y_1\| \leq \|x_1\| < a$.

(b) As seen in the proof of Case (a), the conditions (C2) and (C3) lead to $\|Sx\| \leq \|x\|$ for $x \in C \cap \partial\Omega_a$. Next, since $f_\infty = \infty$, we may choose $w > a$ such that

$$f(u, v) \geq Av, \quad u \geq w,\ v \geq w \tag{3.27}$$

where A is defined in (3.25). Let

$$w_0 = \max\left\{\frac{w}{\gamma},\ \frac{w}{\left(\frac{1}{2} - \delta\right)\gamma}\right\} = \frac{w}{\left(\frac{1}{2} - \delta\right)\gamma}.$$

and $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. Note that $w_0 > w > a$. We shall show that $\|Sx\| \geq \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$. Let $x \in C \cap \partial\Omega_{w_0}$. So $\|x\| = w_0$ and it is clear that

$$x(s) \geq \gamma\|x\| \geq w,\ s \in [\delta, 1-\delta] \qquad \text{and} \qquad \int_\delta^{\frac{1}{2}} x(\tau)d\tau \geq \left(\frac{1}{2} - \delta\right)\gamma\|x\| = w.$$

Using these together with (C4), Lemma 2.2, (3.27) and (3.25), we get for $t \in [\delta, 1-\delta]$,

$$\begin{aligned}
|Sx(t)| &\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)|\alpha(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)|\alpha(s) f\left(\int_\delta^{\frac{1}{2}} x(\tau)d\tau, x(s)\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} |g_m(t,s)|\alpha(s) A x(s) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\, \alpha(s) A\gamma\|x\| \sin \pi s ds \;=\; \|x\|.
\end{aligned}$$

It follows that $\|Sx\| \geq \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$.

Applying Theorem 2.2, we conclude that S has a fixed point $x_2 \in C \cap (\bar{\Omega}_{w_0} \backslash \Omega_a)$ such that $a \leq \|x_2\| \leq w_0$. Once again as seen earlier $\|x_2\| \neq a$, so $a < \|x_2\| \leq w_0$. From (3.4) and (3.18), we have $y_2(t) = \int_0^t x_2(s)ds$ is a positive solution of (1.1) with $\|y_2\| \leq \|x_2\| \leq w_0$ and $y_2(t) \geq \gamma\|x_2\|(t-\delta) > \gamma a(t-\delta)$ for $t \in [\delta, 1-\delta]$.

(c) This follows from Cases (a) and (b). $\square$

Theorem 3.7. Let (C1), (C2), (C4), (C5) hold, and $0 < \int_0^1 \beta(s) \sin \pi s ds < \infty$.

(a) If $f_0 = 0$, then (1.1) has a nontrivial positive solution $y_1 \in C[0,1]$ such that $0 < \|y_1\| \leq b$.

(b) If $f_\infty = 0$, then (1.1) has a nontrivial positive solution $y_2 \in C[0,1]$ such that $y_2(t) \geq \gamma b(t-\delta)$ for $t \in [\delta, 1-\delta]$.

(c) If $f_0 = f_\infty = 0$, then (1.1) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that

$$0 < \|y_1\| \leq b \qquad \text{and} \qquad y_2(t) \geq \gamma b(t-\delta),\ t \in [\delta, 1-\delta].$$

Proof. Once again we shall apply Theorem 2.2 with the cone C defined in (3.11).

(a) Let

$$\tilde{A} = \left[\int_0^1 \frac{1}{\pi^{2m-1}}\, \beta(s) \sin \pi s ds\right]^{-1}. \tag{3.28}$$

Since $f_0 = 0$, there exists $0 < r < b$ such that

$$f(u,v) \leq \tilde{A}v, \quad 0 < u \leq r,\ 0 < v \leq r. \tag{3.29}$$

Let $\Omega_r = \{x \in B \mid \|x\| < r\}$. We shall show that $\|Sx\| \leq \|x\|$ for $x \in C \cap \partial\Omega_r$. To proceed, let $x \in C \cap \partial\Omega_r$. So $\|x\| = r$. Using (C2), Lemma 2.1, (3.29) and (3.28), we find for $t \in [0,1]$,

$$\begin{aligned}
|Sx(t)| &\leq \int_0^1 |g_m(t,s)|\beta(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\
&\leq \int_0^1 |g_m(t,s)|\beta(s)\tilde{A}x(s)ds \\
&\leq \int_0^1 \frac{1}{\pi^{2m-1}}\, \beta(s)\tilde{A}\|x\| \sin \pi s ds \;=\; \|x\|.
\end{aligned}$$

Hence, $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_r$.

Next, let $\Omega_b = \{x \in B \mid \|x\| < b\}$. For $x \in C \cap \partial\Omega_b$, using (C4) and (C5) as in the proof of Theorem 3.3, we obtain (3.16). Thus, $\|Sx\| \ge \|x\|$ for $x \in C \cap \partial\Omega_b$.

It now follows from Theorem 2.2 that S has a fixed point $x_1 \in C \cap (\bar{\Omega}_b \backslash \Omega_r)$ such that $r \le \|x_1\| \le b$. In view of (3.4), the conclusion is clear.

(b) As seen in the proof of Case (a), the conditions (C4) and (C5) lead to $\|Sx\| \ge \|x\|$ for $x \in C \cap \partial\Omega_b$. Next, since $f_\infty = 0$, we may choose $w > b$ such that

$$f(u,v) \le \tilde{A}v, \quad u \ge w,\ v \ge w \tag{3.30}$$

where $\tilde{A}$ is defined in (3.28). We shall consider two cases – when f is bounded and when f is unbounded.

<u>Case 1</u> Suppose that f is bounded. Then, there exists some $Q > 0$ such that

$$f(u,v) \le Q, \quad u, v \in [0,\infty). \tag{3.31}$$

Let

$$w_0 = \max\left\{b+1,\ \frac{Q}{\pi^{2m-1}}\int_0^1 \beta(s)\sin\pi s ds\right\}$$

and $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. For $x \in C \cap \partial\Omega_{w_0}$, using (C2), Lemma 2.1 and (3.31), we get for $t \in [0,1]$,

$$\begin{aligned} |Sx(t)| &\le \int_0^1 |g_m(t,s)|\beta(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\le \int_0^1 \frac{1}{\pi^{2m-1}}\ \beta(s) Q \sin\pi s ds \ \le\ w_0 \ =\ \|x\|. \end{aligned}$$

Hence, $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$.

<u>Case 2</u> Suppose that f is unbounded. Then, there exists $w_0 > w$ $(> b)$ such that

$$f(u,v) \le f(w_0,w_0), \quad 0 \le u \le w_0,\ 0 \le v \le w_0. \tag{3.32}$$

Let $x \in C \cap \partial\Omega_{w_0}$ where $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. Then, applying (C2), Lemma 2.1, (3.32), (3.30) and (3.28) successively gives for $t \in [0,1]$,

$$\begin{aligned} |Sx(t)| &\le \int_0^1 |g_m(t,s)|\beta(s) f\left(\int_0^s x(\tau)d\tau, x(s)\right) ds \\ &\le \int_0^1 |g_m(t,s)|\beta(s) f(w_0,w_0) ds \\ &\le \int_0^1 \frac{1}{\pi^{2m-1}}\ \beta(s)\tilde{A} w_0 \sin\pi s ds \ =\ w_0 \ =\ \|x\|. \end{aligned}$$

Thus, $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$.

Having established $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$ in the above two cases, we can now apply Theorem 2.2 to conclude that S has a fixed point $x_2 \in C \cap (\bar{\Omega}_{w_0} \backslash \Omega_b)$ such that $b \le \|x_2\| \le w_0$. In view of (3.4) and (3.18), the proof is complete.

(c) This is immediate from Cases (a) and (b). $\square$

Our last result gives the existence of <u>*multiple positive*</u> solutions of (1.1).

Theorem 3.8. Assume (C1), (C2) and (C4) hold. Let (C3) be satisfied for $a = a_\ell,\ \ell = 1, 2, \cdots, k$, and (C5) be satisfied for $b = b_\ell,\ \ell = 1, 2, \cdots, n$.

(a) If $n = k + 1$ and $0 < b_1 < a_1 < \cdots < b_k < a_k < b_{k+1}$, then (1.1) has (at least) $2k$ nontrivial positive solutions $y_1, \cdots, y_{2k} \in C[0, 1]$ such that for $\ell = 1, 2, \cdots, k$,

$$\|y_{2\ell-1}\| < a_\ell, \quad \|y_{2\ell}\| \le b_{\ell+1}; \quad y_{2\ell-1}(t) \ge \gamma b_\ell(t-\delta), \quad y_{2\ell}(t) > \gamma a_\ell(t-\delta), \quad t \in [\delta, 1-\delta].$$

(b) If $n = k$ and $0 < b_1 < a_1 < \cdots < b_k < a_k$, then (1.1) has (at least) $2k - 1$ nontrivial positive solutions $y_1, \cdots, y_{2k-1} \in C[0, 1]$ such that for $\ell = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, k - 1$,

$$\|y_{2\ell-1}\| < a_\ell, \quad \|y_{2j}\| \le b_{j+1}; \quad y_{2\ell-1}(t) \ge \gamma b_\ell(t-\delta), \quad y_{2j}(t) > \gamma a_j(t-\delta), \quad t \in [\delta, 1-\delta].$$

(c) If $k = n+1$ and $0 < a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$, then (1.1) has (at least) $2n+1$ positive solutions $y_0, \cdots, y_{2n} \in C[0, 1]$ such that for $\ell = 1, 2, \cdots, n$,

$$\|y_0\| < a_1, \quad \|y_{2\ell-1}\| \le b_\ell, \quad \|y_{2\ell}\| < a_{\ell+1};$$
$$y_{2\ell-1}(t) > \gamma a_\ell(t-\delta), \quad y_{2\ell}(t) \ge \gamma b_\ell(t-\delta), \quad t \in [\delta, 1-\delta].$$

Note that $y_1, \cdots, y_{2n}$ are nontrivial.

(d) If $k = n$ and $0 < a_1 < b_1 < \cdots < a_k < b_k$, then (1.1) has (at least) $2k$ positive solutions $y_0, \cdots, y_{2k-1} \in C[0, 1]$ such that for $\ell = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, k - 1$,

$$\|y_0\| < a_1, \quad \|y_{2\ell-1}\| \le b_\ell, \quad \|y_{2j}\| < a_{j+1};$$
$$y_{2\ell-1}(t) > \gamma a_\ell(t-\delta), \quad y_{2j}(t) \ge \gamma b_j(t-\delta), \quad t \in [\delta, 1-\delta].$$

Note that $y_1, \cdots, y_{2k-1}$ are nontrivial.

Proof. In (a) and (b), we just apply (3.17) (in the proof of Theorem 3.3) repeatedly to get multiple positive solutions of (3.3) as follows.

(a) If $n = k + 1$ and $0 < b_1 < a_1 < \cdots < b_k < a_k < b_{k+1}$, then (3.3) has (at least) $2k$ nontrivial positive solutions $x_1, \cdots, x_{2k} \in C[0, 1]$ such that

$$0 < b_1 \le \|x_1\| < a_1 < \|x_2\| \le b_2 \le \cdots < a_k < \|x_{2k}\| \le b_{k+1}.$$

(b) If $n = k$ and $0 < b_1 < a_1 < \cdots < b_k < a_k$, then (3.3) has (at least) $2k - 1$ nontrivial positive solutions $x_1, \cdots, x_{2k-1} \in C[0, 1]$ such that

$$0 < b_1 \le \|x_1\| < a_1 < \|x_2\| \le b_2 \le \cdots \le b_k \le \|x_{2k-1}\| < a_k.$$

In (c) and (d), from the proof of Theorem 3.2 we obtain the existence of a positive solution x_0 of (3.3) with $0 \le \|x_0\| < a_1$, then we apply (3.17) repeatedly to get other positive solutions of (3.3) as follows.

(c) If $k = n+1$ and $0 < a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$, then (3.3) has (at least) $2n+1$ positive solutions $x_0, \cdots, x_{2n} \in C[0,1]$ such that

$$0 \le \|x_0\| < a_1 < \|x_1\| \le b_1 \le \|x_2\| < a_2 < \cdots \le b_n \le \|x_{2n}\| < a_{n+1}.$$

(d) If $k = n$ and $0 < a_1 < b_1 < \cdots < a_k < b_k$, then (3.3) has (at least) $2k$ positive solutions $x_0, \cdots, x_{2k-1} \in C[0,1]$ such that

$$0 \le \|x_0\| < a_1 < \|x_1\| \le b_1 \le \|x_2\| < a_2 < \cdots < a_k < \|x_{2k-1}\| \le b_k.$$

The proof is complete by using (3.4) and (3.18). $\square$

Remark 3.4. In view of Remark 3.2, the conditions (C4) and (C5) in Theorems 3.4, 3.5, 3.7 and 3.8 may be replaced by (C4)$'$ and (C5)$'$. Note, however, that (C4) in Theorem 3.6 *cannot* be replaced by (C4)$'$.

We shall now illustrate the results obtained by some examples.

Example 3.1. Consider the complementary Lidstone boundary value problem

$$\begin{gathered} y^{(5)} = F(t,y,y') = 24\left(\frac{t^5}{10} + \frac{t^4}{4} - t^3 + \frac{t^2}{4} + \frac{t}{2} + 2\right)^{-2}\left[\frac{y+y'}{2} + 2\right]^2, \quad t \in [0,1] \\ y(0) = y'(0) = y'''(0) = y'(1) = y'''(1) = 0. \end{gathered} \tag{3.33}$$

Here, $m = 2$. Let $\delta = \frac{1}{4}$. So $\gamma = \frac{1}{2\pi}$. Clearly, (C1) is satisfied. Further, (C2) and (C4) are fulfilled if we choose

$$\alpha(t) = \beta(t) = 24\left(\frac{t^5}{10} + \frac{t^4}{4} - t^3 + \frac{t^2}{4} + \frac{t}{2} + 2\right)^{-2} \qquad \text{and} \qquad f(u,v) = \left(\frac{u+v}{2} + 2\right)^2.$$

Next, in view of Remark 3.3 (see (3.19)), (C3) is satisfied provided

$$a > f(a,a)\int_0^1 \frac{1}{\pi^{2m-1}}\ \beta(s)\sin \pi s ds = (a+2)^2\int_0^1 \frac{1}{\pi^3}\ \beta(s)\sin \pi s ds$$

which is solved to get $a \in [0.8467, 4.7247]$.

Hence, (C1)–(C4) are met and also $f_0 = f_\infty = \infty$. We conclude from Theorem 3.6 that (3.33) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that

$$0 < \|y_1\| < a \qquad \text{and} \qquad y_2(t) > \frac{1}{2\pi}\ a\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Since $a \in [0.8467, 4.7247]$, it follows that

$$0 < \|y_1\| < 0.8467 \qquad \text{and} \qquad y_2(t) > \frac{1}{2\pi}(4.7247)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.34}$$

In fact, by direct computation a positive solution of (3.33) is given by $y^*(t) = \frac{t^5}{5} - \frac{t^4}{2} + \frac{t^2}{2}$ with

$$\|y^*\| = 0.2 \qquad \text{and} \qquad y^*(t) \ge \frac{1}{2\pi}(2.1426)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.35}$$

(Note that the number 2.1426 in (3.35) is the <u>*largest*</u> c for the inequality $y^*(t) \geq \frac{1}{2\pi}c\left(t-\frac{1}{4}\right)$ to hold for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$.) This y^* validates the conclusion (3.34) about y_1, this y^* is not y_2.

Example 3.2. Consider the complementary Lidstone boundary value problem

$$\begin{gathered} y^{(5)} = F(t, y, y') = 24\left(\frac{t^5}{10} + \frac{t^4}{4} - t^3 + \frac{t^2}{4} + \frac{t}{2} + 2\right)^{-q}\left[\frac{y+y'}{2} + 2\right]^q, \quad t \in [0,1] \\ y(0) = y'(0) = y'''(0) = y'(1) = y'''(1) = 0 \end{gathered} \tag{3.36}$$

where $q > 0$.

Once again let $\delta = \frac{1}{4}$. So $\gamma = \frac{1}{2\pi}$. Clearly, (C1) is satisfied. Further, (C2) and (C4) are fulfilled if we choose

$$\alpha(t) = \beta(t) = 24\left(\frac{t^5}{10} + \frac{t^4}{4} - t^3 + \frac{t^2}{4} + \frac{t}{2} + 2\right)^{-q} \qquad \text{and} \qquad f(u,v) = \left(\frac{u+v}{2} + 2\right)^q.$$

Next, noting Remark 3.3 (see (3.19) and (3.20)), (C3) and (C5) are satisfied provided

$$a > f(a,a)\int_0^1 \frac{1}{\pi^{2m-1}}\,\beta(s)\sin \pi s ds = (a+2)^q \int_0^1 \frac{1}{\pi^3}\,\beta(s)\sin \pi s ds \tag{3.37}$$

and

$$b \leq f\left(\gamma b\left(\frac{1}{2} - \delta\right), \gamma b\right)\int_{\frac{1}{2}}^{1-\delta} \frac{2\delta}{\pi^{2m}}\,\alpha(s)\sin \pi s ds = \left(\frac{5b}{16\pi} + 2\right)^q \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{2\pi^4}\,\alpha(s)\sin \pi s ds. \tag{3.38}$$

Solving (3.37) and (3.38) for different values of q gives the following ranges of a and b.

q	(C3) is satisfied if	(C5) is satisfied if
$\frac{1}{2}$	$a \in [0.5320, \infty)$	$b \in (0, 0.0263]$
1	$a \in [0.5869, \infty)$	$b \in (0, 0.0251]$
2	$a \in [0.8467, 4.7247]$	$b \in (0, 0.0227] \cup [17760.50, \infty)$

Hence, (C1)–(C5) are fulfilled.

<u>Case 1</u> $q = \frac{1}{2}$. From the above table, we see that $a > b$. By Theorem 3.3(b), we conclude that (3.36) has a nontrivial positive solution $y^* \in C[0,1]$ such that $\|y^*\| < a$ and $y^*(t) \geq \gamma b(t-\delta)$ for $t \in [\delta, 1-\delta]$. Noting the ranges of a and b, we further obtain

$$\|y^*\| < 0.5320 \qquad \text{and} \qquad y^*(t) \geq \frac{1}{2\pi}(0.0263)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.39}$$

<u>Case 2</u> $q = 1$. Once again we have $a > b$. Hence, using Theorem 3.3(b) and the ranges of a and b, we see that (3.36) has a nontrivial positive solution $y^* \in C[0,1]$ such that

$$\|y^*\| < 0.5869 \qquad \text{and} \qquad y^*(t) \geq \frac{1}{2\pi}(0.0251)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.40}$$

Case 3 $q = 2$. Applying Theorem 3.5 with $\tilde{b} \in (0, 0.0227]$ and $b \in [17760.50, \infty)$, we see that (3.36) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that $\|y_1\| < a$, $\|y_2\| \le b$ and $y_1(t) \ge \gamma\tilde{b}(t-\delta)$, $y_2(t) > \gamma a(t-\delta)$, $t \in [\delta, 1-\delta]$. In view of the ranges of $\tilde{b}$, a and b, we further conclude that

$$\begin{gathered} \|y_1\| < 0.8467, \quad \|y_2\| \le 17760.50; \\ y_1(t) \ge \frac{1}{2\pi}(0.0227)\left(t - \frac{1}{4}\right), \quad y_2(t) > \frac{1}{2\pi}(4.7247)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \end{gathered} \tag{3.41}$$

Note that by direct computation a positive solution of (3.36) is given by $y^*(t) = \frac{t^5}{5} - \frac{t^4}{2} + \frac{t^2}{2}$ such that (3.35) holds. Clearly, this y^* validates the conclusions (3.39) and (3.40). This y^* may be y_1 but certainly not y_2 in (3.41).

Remark 3.5. The boundary value problem (3.33) is actually (3.36) when $q = 2$. We see that the conclusion (3.41) (obtained from Theorem 3.5) gives more details than the conclusion (3.34) (obtained from Theorem 3.6). Note that the condition (C5) is required in Theorem 3.5 but not in Theorem 3.6, and it takes more effort to check (C5). The 'more' details in (3.41) come at the expense of a comparatively more complex condition.

Remark 3.6. In Example 3.2, (C4)$'$ is also satisfied with $\alpha_0 = \alpha$. Moreover, (C5)$'$ is fulfilled provided (see (3.21))

$$b \le f\left(\gamma b(1-2\delta), \gamma b\right) \int_{1-\delta}^{1} \frac{2\delta}{\pi^{2m}}\, \alpha_0(s) \sin \pi s ds = \left(\frac{3b}{8\pi} + 2\right)^q \int_{\frac{3}{4}}^{1} \frac{1}{2\pi^4}\, \alpha_0(s) \sin \pi s ds. \tag{3.42}$$

Solving (3.42) for the same values of q as in Example 3.2 gives the following *new* ranges of b.

q	(C3) is satisfied if	(C5)$'$ is satisfied if
$\frac{1}{2}$	$a \in [0.5320, \infty)$	$b \in (0, 0.0110]$
1	$a \in [0.5869, \infty)$	$b \in (0, 0.0105]$
2	$a \in [0.8467, 4.7247]$	$b \in (0, 0.0097] \cup [28722.08, \infty)$

Hence, (C1)–(C3), (C4)$'$ and (C5)$'$ are fulfilled.

Case 1 $q = \frac{1}{2}$. We have $a > b$. Using Theorem 3.3(b) and the ranges of a and b, we conclude that (3.36) has a nontrivial positive solution $y^* \in C[0,1]$ such that

$$\|y^*\| < 0.5320 \qquad \text{and} \qquad y^*(t) \ge \frac{1}{2\pi}(0.0110)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.39$'$}$$

Case 2 $q = 1$. Once again we have $a > b$. Applying Theorem 3.3(b) again, we see that (3.36) has a nontrivial positive solution $y^* \in C[0,1]$ such that

$$\|y^*\| < 0.5869 \qquad \text{and} \qquad y^*(t) \ge \frac{1}{2\pi}(0.0105)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.40$'$}$$

Case 3 $q = 2$. Using Theorem 3.5 with $\tilde{b} \in (0, 0.0097]$ and $b \in [28722.08, \infty)$, we see that (3.36) has (at least) two nontrivial positive solutions $y_1, y_2 \in C[0,1]$ such that

$$\|y_1\| < 0.8467, \quad \|y_2\| \le 28722.08;$$
$$y_1(t) \ge \frac{1}{2\pi}(0.0097)\left(t - \frac{1}{4}\right), \quad y_2(t) > \frac{1}{2\pi}(4.7247)\left(t - \frac{1}{4}\right), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.41$'$}$$

It would *appear* that (3.39)–(3.41) are 'sharper' than (3.39)′–(3.41)′. However, it should be noted that all the theorems in this paper give existence of at least one or two or multiple solutions. Hence, the solutions in (3.39)–(3.41) may be *different* from the solutions in (3.39)′–(3.41)′. So we cannot really compare (3.39)–(3.41) and (3.39)′–(3.41)′. From these conclusions we do get more information about the solutions of the boundary value problem (3.36).

Once again a known positive solution of (3.36), $y^*(t) = \frac{t^5}{5} - \frac{t^4}{2} + \frac{t^2}{2}$ (with (3.35)), validates the conclusions (3.39)′ and (3.40)′, and this y^* may be y_1 but certainly not y_2 in (3.41)′.

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